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OBSERVATIONS ON THE MEASURE OF CHANGE.

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It is clear that one who attempts to study precisely things that are changing must have a great deal to do with measures of change. Now, almost all those phenomena of society with which the statistician is chiefly interested are in constant motion, cannot be caught and pinned down permanently in one place, but must be taken on the wing and their velocity measured. Next to their present position their direction and velocity are the most important things to be known about them, since these alone give us any power to forecast future positions.

It is to be noted in regard to measures of change that, like all other measures, their usefulness depends very much on the degree in which they facilitate comparisons. As we use feet and inches to get a precise notion of the relation between two lengths, so we need a standard measure that shall enable us accurately to compare the magnitude of various changes. A rude measure of change is obtained by a simple statement

of the difference between the two states of the quantity studied, as when we say that the population of the United States increased some twelve millions during the last decade. But this does not enable us to tell whether the increase in the length of railways, supposing that to be known, kept pace with the population or not; does not even answer satisfactorily the question how the increase of negroes, some 885 thousand, compares with that of the population as a whole. We want to be able, if possible, to say this change is equal to that; this is twice as great or a half greater; this the same but in the opposite direction, etc.

A measure intended to supply this need is actually in use in statistical work. This is the per cent of increase, or, more accurately, the ratio between two stages of a quantity. The colored population, it would appear, increased only about 13 in the hundred, as against 25 in the hundred for the population in general. Ohio grew about one per cent faster than New York, and about seven per cent slower than Pennsylvania,— comparisons impossible without some standard measure of change. Clearly the idea here is that of finding the relation between the magnitude of the change and that of the quantity changing, and using this relation as a means of comparison. "For its size," Detroit grew much more than New York, though less than Chicago.

I take it that, leaving aside a very few thoughtful students, this relation computed in the ordinary way is accepted as being in truth what it apparently purports to be, an absolute and true measure of change. An increase of 50 per cent is twice as great as an increase of 25 per cent, and equal, though differently directed, to a decrease of 50 per cent. In the ordinary statistical presentation of such facts there is everything to corroborate this notion, and nothing to warn against it.

Yet surely it may readily be shown that the notion is altogether erroneous. Suppose, for example, that during the past ten years a certain city changed in population from 20

thousand to 40 thousand, while another changed from 40 thousand to 20 thousand. It is axiomatic that these changes are equal though of opposite direction; are negative equivalents. Occurring successively in the same city, they would bring it back precisely to its first state. Yet, in the column headed "per cent of increase or decrease" these changes would appear as an increase of 100 per cent, and a decrease of 50 per cent, respectively. If prices were the subject of study, changes of from 20 cents to 40 and the reverse would be expressed in the same way. If these per cents were now added together and averaged, as is commonly done in this latter class of researches, we should have for the two prices an average increase of 25 per cent,—an absurdity sufficient to cast the darkest suspicion on the whole process. Changes from 20 to 30 cents and back will yield an average increase of over 8 per cent, another contradiction of the axiom *ex nihilo nihil fit.* As we have in each case a mean increase, one may suspect at once that this method makes increases appear greater, as compared with decreases, than they really are.

Such inconsistencies as these have by no means passed unobserved, though from the small recognition they receive in statistical practice, it would appear that the perception of them is not at all general. Prof. Jevons, as is well known, favored and practiced the treatment of such ratios by the method of geometric means. Before speaking of this method, however, I wish to apply the test used above to the method of harmonic means, which has also been proposed. In the first number of the *Quarterly Journal of Economics*, page 83, Mr. F. Coggeshall presents some acute criticisms upon the method of Jevons, shows that the harmonic mean produces results in some respects less objectionable, and seems to suggest the use of the latter wherever the greater difficulty of computation does not preclude it. Suppose, however, we test the harmonic mean by the axiom already suggested, that

a true measure of change must show the equality of equal changes. The formula for this mean is —

$$\frac{1}{x} = \frac{\frac{1}{a} + \frac{1}{b} + \cdots \frac{1}{m}}{n}$$

where a, b, etc. are the ratios or per cents to be averaged, n their number, and x the required mean. Applied to the case of two prices mentioned above, undergoing equal but opposite changes, this method would show, where the changes are from 20 to 40 cents and back, a mean decrease of 20 per cent. Equal changes of 10 cents give a mean decrease of nearly 8 per cent. Instead of the mean condition remaining unchanged, as is actually the case, it is represented as falling from 100 to 80 in the first case, or 92 in the second.

As I have never seen the question treated from precisely this point of view, I venture to offer a tabulated statement of the effect of treating pairs of equal changes by these two methods.

A TEST OF MEANS BY PAIRS OF EQUAL CHANGES.

Equal Changes.	Common Expression of Changes.	Arithmetic Mean.		Harmonic Mean.	
		Mean.	Error in Mean.	Mean.	Error in Mean.
20–21 21-20	100-105 100-95.2	100.1	+.1	99.9	—.1
20–22 22–20	100-110 100-90.9	100.5	+.5	99.5	—.5
20–24 24–20	100-120 100-83.3	101.7	+1.7	98.4	—1.6
20-26 26-20	100-130 100-76.9	103.5	+3.5	96.6	—3.4
20-30 30-20	100-150 100-66.7	108.3	+8.3	92.3	—7.7
20-40 40-20	100-200 100-50	125.0	+25.0	80.0	—20.0
20-60 60-20	100-300 100-33.3	166.7	+66.7	60.0	—40.0

From the point of view here taken it certainly seems that one might draw from these facts two propositions, namely,—

1. That the common method of averaging ratios or index numbers by the arithmetical mean exaggerates increases relatively to decreases; and this so much that it is unfit for use, except where the degrees of change are minute.*

2. That the method of the harmonic mean exaggerates decreases relatively to increases in an almost equal degree, and is similarly unfit for use.

The use of the geometric mean in connection with these ratios or index numbers is not so readily assailable. This mean, of course, is that obtained by multiplying together the several quantities to be averaged, and taking that root of their product whose index is the number of quantities. Thus, if there are three quantities the cube root of their product is taken. The use of logarithms makes this process nearly as easy as that for the arithmetic mean. Applied to pairs of equal changes it will always justify itself by showing no change in the mean,— as may readily be ascertained by averaging in this manner the ratios in the second column of the table. This, to be sure, does not prove the method correct, but only that its errors, if there are any, enter equally into increases and decreases. It does, however, make a strong probability in its favor. Indeed, one may say, without pretending to great mathematical insight, that it seems reasonable that the mean force of a number of quantities *as multipliers* (ratios being such) is obtained by that method which averages them as multipliers, that is, by the method of geometric means.

But there is certainly something unsatisfactory in the results of the geometric mean. Mr. Coggeshall perceived

* Unless I am mistaken, even Prof. Falkner has been misled by the use of this method. On p. lxviii of his recent admirable Report to the Senate Committee on Finance I read: "The result of such a calculation [for a mean of index numbers] * * * * is 100.31 as the ratio for these special industries. This is a confirmation of the results," etc.; the ratio being taken as showing a slight mean increase. Of course, this increase is unimportant in any case, but is it not also entirely fictitious? I find the geometrical [true ?] mean of these ratios to be 100.00+. *It is the inevitable and persistent tendency of this method,* even in the ablest hands, *to make increases appear where none exist, and even to change real decreases into apparent increases.*

this when he wrote the criticism above mentioned, but it does not seem to me that he indicated the true nature of the difficulty. I imagine that I can suggest where this difficulty lies, even though I cannot propose a satisfactory remedy.

Does not the trouble arise from the confusion of two distinct questions? These are, I think, first, What is the proper way of obtaining the mean of a series of ratios? and, second, What is the best way to measure changes? To the first question I should answer that the use of the geometric mean affords the only correct method of averaging ratios (or index numbers, which are nothing different), and for proof of this I would point to the considerations that have already been brought forward, and to many others that might be. To the second I should answer that the use of ratios for the measurement and comparison of changes, though mathematically justifiable, is open to serious practical objections. These are the difficulty of understanding and the inconvenience of applying the methods by which alone these ratios can correctly be compared with one another. As already suggested, an increase of 50 per cent is not at all the negative equivalent of a decrease of 50 per cent, nor has it twice the value of an increase of 25 per cent. These relations are inevitably suggested to the unthinking mind by the usual statistical presentation of facts of change, especially by those decapitated quantities contained in columns headed "Per cents of increase or decrease." But no such relations exist. The change negatively equivalent to an increase of 50 per cent is expressed by that ratio which, multiplied by 1.50, will produce unity, namely, .67. The ratio expressing a change having half the value of an increase of 50 per cent is found only by extracting the square root of 1.50. As a means of exposition, then, of expressing to the ordinary mind the comparative value of changes, this method is open to the gravest objection.

If a way could be devised by which these comparisons could be made in the simple and direct manner that we use

in comparing measurements of length or weight, it would, I think, be of the greatest usefulness, and do away with much misunderstanding.

I have myself made some use of a method of denoting the relative value of changes which, though not invulnerable on the mathematical side, renders possible a more simple and direct comparison. That it can be made widely useful I would not venture to assert. It is reached in the following manner.

It has already been suggested that the idea underlying the measurement of changes by per cents is that of finding the relation between the magnitude of the change and that of the quantity changing. In the ordinary method this relation is expressed as the per cent giving the magnitude of the change as compared with the first state of the quantity. But why the first state? Why is a change of from 100 to 150 a change of 50 per cent rather than of 33.3 per cent? There seems no reason in the nature of things why the first state should be chosen as a base rather than the last. The quantity in question has varied, according to some unknown law, from 100 to 150. If a single number is to be chosen as a representative of its magnitude during this period, surely that number should be neither of the extremes, but one representing the mean or average condition,— say 125. The method, then, is to base the per cent of change not on the first state of the changing quantity but on the mean between the two states. The following advantages may be claimed for this plan: —

Equal increases and decreases will be represented by the same per cents. A decrease of 50 per cent will be in fact the negative equivalent of an increase of 50 per cent. Thus, a change from 100 to 200 will be represented by +.67, the reverse by —.67. An immediate and exact ascertainment of this relation of equality, the most important of all, will be possible.

The true relation among various changes will be approxi-

mately represented by the simple arithmetical relation among the per cents of change. The difference between the two methods in this respect may be shown as follows.

The change from 100 to 200 may be regarded as the sum of the following changes: 100–114, 114–136, 136–180, 180–200. These would be expressed by the following per cents:

Change.		Common Expression.	Proposed Expression.
100-114		14 per cent.	13.1 per cent.
114-136		19.3 "	17.6 "
136-180		32.4 "	27.8 "
180-200		11.1 "	10.5 "
	Sum,	76.8	69.0
100-200		100.0	66.6

The point here brought out is that while under the common method the total change appears as something very different from the sum of all its parts, the discrepancy under the method suggested is quite small.